W9. Calculate

$$\sum_{n=2}^{\infty} \frac{H_n H_{n+1}}{(n-1)n}$$

where $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ denotes the n –th harmonic number.

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Note that $\frac{H_n H_{n+1}}{(n-1)n} = \frac{H_n H_{n+1}}{n-1} - \frac{H_n H_{n+1}}{n} = \frac{H_n\left(H_n + \frac{1}{n+1}\right)}{n-1} - \frac{\left(H_{n+1} - \frac{1}{n+1}\right)H_{n+1}}{n} =$
$\frac{H_n^2}{n-1} - \frac{H_{n+1}^2}{n} + \frac{H_n}{(n-1)(n+1)} + \frac{H_{n+1}}{n(n+1)}$.

Since $\frac{H_n}{(n-1)(n+1)} = \frac{1}{2}\left(\frac{H_n}{n-1} - \frac{H_n}{n+1}\right) = \frac{1}{2}\left(\frac{H_{n-1} + \frac{1}{n}}{n-1} - \frac{H_{n+1} - \frac{1}{n+1}}{n+1}\right) =$
$\frac{1}{2}\left(\frac{H_{n-1}}{n-1} - \frac{H_{n+1}}{n+1}\right) + \frac{1}{2(n-1)n} + \frac{1}{2(n+1)^2} = \frac{1}{2}\left(\frac{H_{n-1}}{n-1} + \frac{H_n}{n} - \left(\frac{H_n}{n} + \frac{H_{n+1}}{n+1}\right)\right) +$
$\frac{1}{2(n-1)n} + \frac{1}{2(n+1)^2}$ and $\frac{H_{n+1}}{n(n+1)} = \frac{H_{n+1}}{n} - \frac{H_{n+1}}{n+1} = \frac{H_n + \frac{1}{n+1}}{n} - \frac{H_{n+1}}{n+1} =$
$\frac{H_n}{n} - \frac{H_{n+1}}{n+1} + \frac{1}{n(n+1)}$ then

$$\frac{H_n H_{n+1}}{(n-1)n} = \left(\frac{H_n^2}{n-1} + \frac{H_{n-1}}{2(n-1)} + \frac{3H_n}{2n}\right) - \left(\frac{H_{n+1}^2}{n} + \frac{H_n}{2n} + \frac{3H_{n+1}}{2(n+1)}\right) + \frac{1}{2(n-1)n} + \frac{1}{2(n+1)^2}$$

and, therefore,

$$\sum_{n=2}^{\infty} \frac{H_n H_{n+1}}{(n-1)n} = \frac{H_2^2}{2-1} + \frac{H_1}{2(2-1)} + \frac{3H_2}{2 \cdot 2} + \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{(n-1)n} + \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{(n+1)^2} =$$

$$\left(\frac{3}{2}\right)^2 + \frac{1}{2} + \frac{3 \cdot \frac{3}{2}}{4} + \frac{1}{2} + \frac{1}{2}\left(\zeta(2) - 1 - \frac{1}{2^2}\right) = \frac{1}{12}\pi^2 + \frac{15}{4}.$$